Connectivity of random irrigation networks

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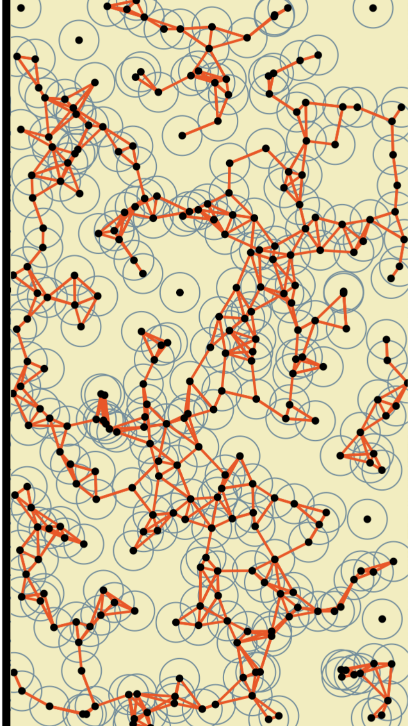

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We consider the case when the underlying graph is a *random geometric graph* such that the vertices are $X_1, \ldots, X_n$ be i.i.d. uniform on $[0,1]^d$ and $X_i \sim X_j$ iff $\|X_i - X_j\| < r$.

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Main question: How large does c need to be for $G(n, r, c)$ to be connected?

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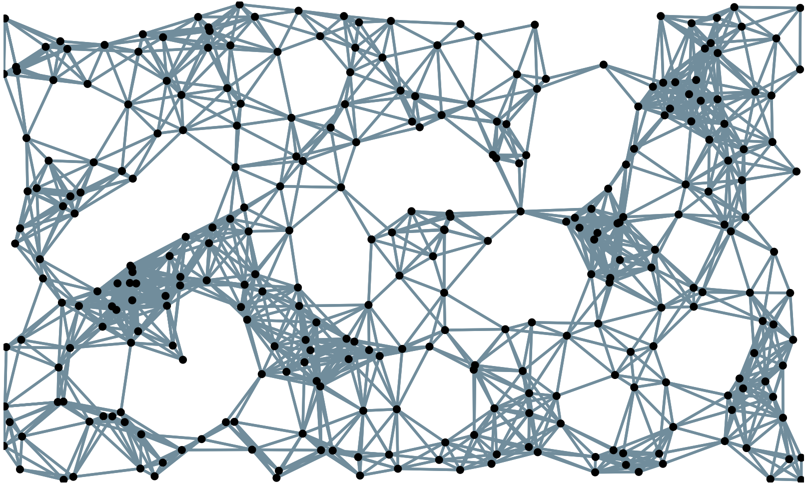
- Penrose (1997) showed that $\forall\,\varepsilon > 0$, $G(n,r)$ is connected whp if $r \geq (1+\varepsilon)r_t$ where

$$r_t = \theta_d \left(\frac{\log n}{n}\right)^{1/d} \qquad \text{and} \qquad \theta_d = \frac{2}{(2d \operatorname{Vol} B(0,1))^{1/d}}.$$

  We only consider values of r above this level.

Irrigation graph



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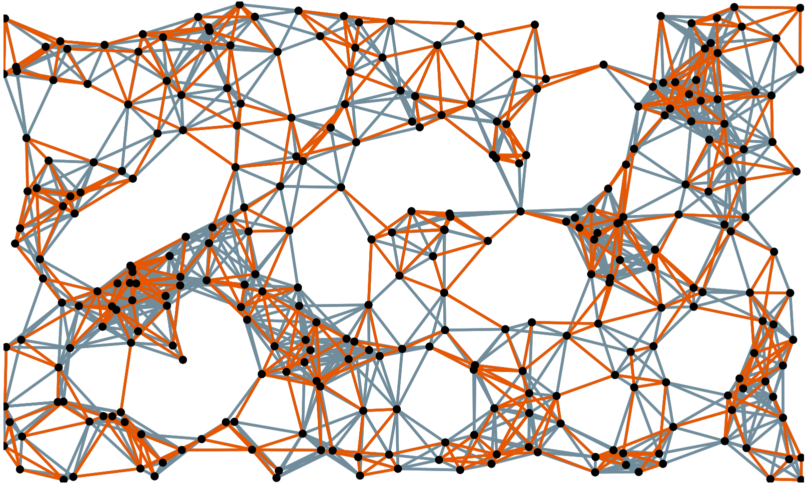

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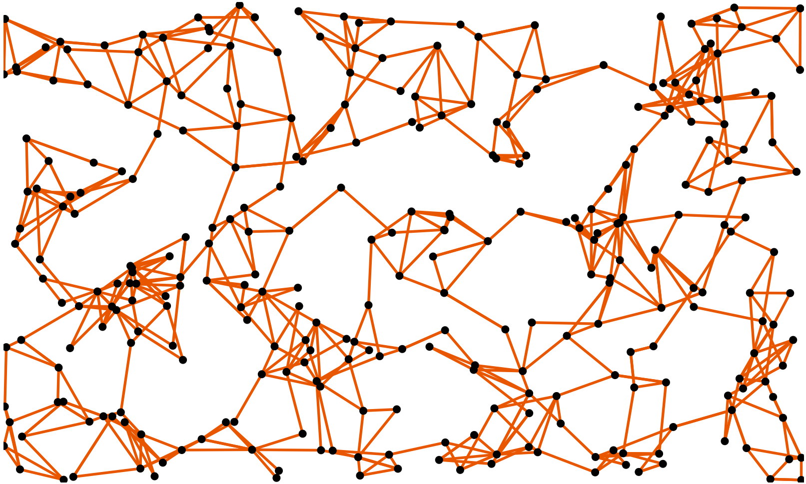

Previous results

Theorem (Fenner and Frieze, 1982)

For $r = \infty$, the graph $G(n, r, 2)$ (the random 2-out graph) is connected whp.

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For constant r the graph $G(n, r, 2)$ is connected whp.

Theorem (Crescenzi, Nocentini, Pietracaprina, Pucci, 2009)

In dimension $d = 2$, $\exists\, \alpha, \beta$ such that if

$$r \geq \alpha \sqrt{\frac{\log n}{n}} \qquad \text{and} \qquad c \geq \beta \log(1/r),$$

then $G(n, r, c)$ is connected whp.

Main result

Theorem

There exists a constant $\gamma^* > 0$ *such that for all* $\gamma \geq \gamma^*$ *and* $\varepsilon \in (0,1)$*, if*

$$r \sim \gamma \left(\frac{\log n}{n} \right)^{1/d} \qquad \textit{and} \qquad c_t = \sqrt{\frac{2 \log n}{\log \log n}},$$

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- *if $c \geq (1+\varepsilon) c_t$ then $G(n,r,c)$ is connected whp.*
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c_t does not depend on γ or d.

Below the threshold

Theorem

Let $\gamma \geq \gamma^*$ *and* $\varepsilon \in (0,1)$*. If* $r = \gamma \left(\frac{\log n}{n}\right)^{1/d}$ *and* $c \leq (1-\varepsilon) c_t$ *then* $G(n,r,c)$ *is disconnected whp.*

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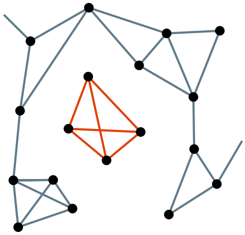

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- Let $\mathscr{F}$ be the random family of subsets of $\{1,\ldots,n\}$ given by

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- Let $I(Q)$ be the indicator of the event that Q is an isolated clique. Then $N = \sum_{Q\in\mathscr{F}} I(Q)$ is the number of isolated $(c+1)$-cliques.

Isolated (c+1)-cliques

- We need some regularity on the uniformly distributed points. For every $1 \leq j \leq n$

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- Let D be the event described above. We use the second-moment method and prove that

$$\mathbf{P}\{N 1_D > 0\} \ge \frac{\mathbf{E}\{N 1_D\}^2}{\mathbf{E}\left\{N^2 1_D\right\}} \to 1.$$

Above the threshold

Theorem

Let $\gamma \geq \gamma^*$ *and* $\varepsilon \in (0,1)$. *If* $r = \gamma \left(\frac{\log n}{n}\right)^{1/d}$ *and* $c \geq (1+\varepsilon)c_t$ *then* $G(n,r,c)$ *is connected whp.*

Gridding and percolation

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- Two cells are connected if they are adjacent and there is an edge between one vertex of each cell.
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- A cell is colored black if all the vertices in it are connected to each other without using an edge that leaves the cell. The other cells are initially colored white.

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4. Every connected component of G has size at least $s = \exp((\log n)^{1/3})$.

The four properties

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- Concentration of number of points in cells.
- $\mathbf{E}\{\# C\} = \Theta(nr^2) = \Theta(\log n)$.

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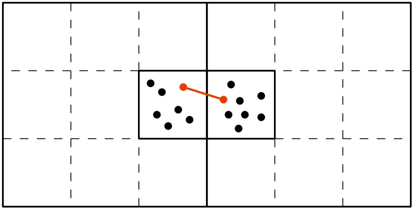
- Subdivide the cell and find an edge bewteen two squares in the border.



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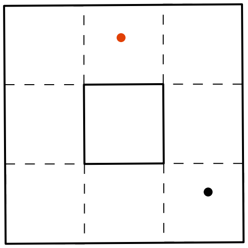

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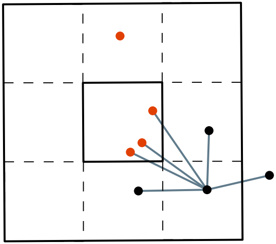

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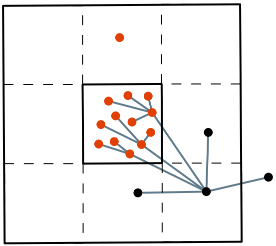

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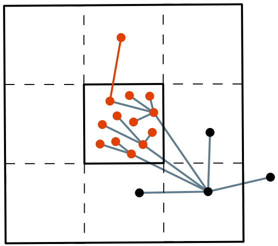

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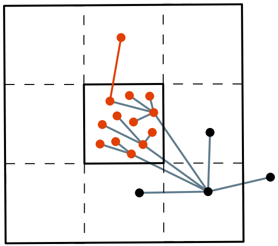

- $\#\{*\text{-connected comp. of size } k\} \le n(8e)^k$.
- It suffices to show that $\mathbf{P}\{\text{Cell is white}\} \le p = \exp(-(\log n)^{2/3})$.
- If $k > q$ then $n(8e)^k p^k \to 0$.

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- Save $(\varepsilon/2)c_t$ edge choices.
- No small components with $(1+\varepsilon/2)c_t$ choices.
- Use extra edges iteratively to double the size of components.

Gridding and percolation

1. Every cell in the grid contains at most $\lambda \log n$ vertices.
2. Every cell in the grid connects to its adjacent cells.
3. Every $*$-connected component of white cells has size at most q cells.
4. Every connected component of G has size at least s.

If all properties hold, then the whole graph is connected.

Everything is connected

- Black connector: There exists a connected component of black cells that links two opposite sides of $[0,1]^2$.

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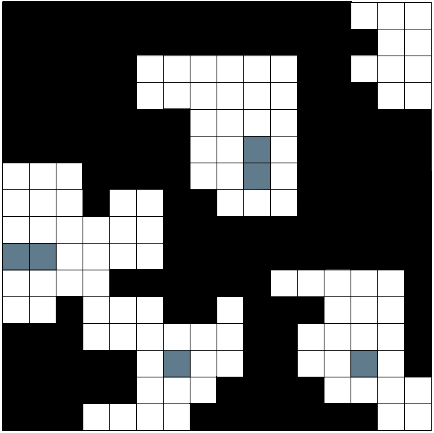
- Black connector: There exists a connected component of black cells that links two opposite sides of $[0,1]^2$.
- Black giant: Black components of size less than $1/r$ are now recolored gray. All remaining black cells are connected. The corresponding vertices of G belong to the same connected component.
- Connectivity: Each vertex connects to at least one vertex of the black giant.

Everything is connected

K



K'

Spanning ratio and diameter

An important feature of a geometric graph is the *spanning ratio*

$$\sup_{i,j} \frac{\mathrm{dist}(X_i, X_j)}{\|X_i - X_j\|}$$

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Unfortunately, this can be large if X_i and X_j are very close.
However, for c slightly larger than critical, we have

Theorem

$\exists\, K, \mu > 0$ *such that if* $\gamma > \gamma^*$, $\quad r = \gamma \left(\frac{\log n}{n}\right)^{1/d}$ *and* $c \geq \mu\sqrt{\log n}$ *then*

$$\sup_{i,j : \|X_i - X_j\| > r} \frac{\mathrm{dist}(X_i, X_j)}{\|X_i - X_j\|} \leq K, \quad \textit{whp.}$$

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Idea of the proof: Partition the unit cube into a grid of cells of side length $\ell = (1/3)\lfloor 1/r \rfloor^{-1}$.

With high probability, any two points i and j, such that X_i and X_j fall in the same cell, are connected by a path of length at most five.

On the other hand, with high probability, any two neighboring cells contain two points, one in each cell, that are connected by an edge of S_n.

Supercritical radii

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Theorem

Let $\varepsilon \in (0,1)$ *and* $\lambda \in [1,\infty]$ *be such that* $\gamma^* \left(\frac{\log n}{n}\right)^{1/d} < r < \sqrt{d}$,

$$\frac{\log nr^d}{\log \log n} \to \lambda \quad \text{and} \quad c \leq (1-\varepsilon)\sqrt{\left(\frac{\lambda}{\lambda - 1/2}\right)\frac{\log n}{\log nr^d}}.$$

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Then $G(n,r,c)$ *is disconnected whp.*

In particular, take $r \sim n^{-(1-\delta)/d}$. Then for $c \le (1-\epsilon)/\sqrt{\delta}$ (constant) the graph is disconnected.

Supercritical r, constant c

We can show that the lower bound is not far from the truth: when $r \sim n^{-(1-\delta)/d}$, constant c is sufficient for connectivity.

$c = \sqrt{5/\delta} + c(d)$ is sufficient for connectivity.

The irrigation graph is genuinely sparse.

Supercritical r, constant c

Theorem

Let $\delta \in (0,1)$, $\gamma > 0$. Suppose that $r_n \sim \gamma n^{-(1-\delta)/d}$. There exists a constant $c = c(\delta, d)$ such that G is connected whp. One may take $c = c_1 + c_2 + c_3 + 1$, where

$$c_1 = \lceil \sqrt{5/(\delta - \delta^2)} \rceil ,$$

and c_2, c_3 depend on d only.

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• We add edges in four phases. In the first we start from X_1, and using c_1 choices of each vertex, we go for $\delta^2 \log_{c_1} n$ generations. There exists a cube in the grid that contains a connected component of size $n^{\text{const.}\delta^2}$.

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• Finally, add just one more connection per vertex so that the entire graph becomes connected.

Thank you